$u_1\Lambda$, respectively, with the damped system. Hence, the damped system can be solved by Holzer's Table 1, which will be called the *simple* (damped) Holzer table (because there is no imaginary number j in its heading) and for which the parameter of the table is necessarily the eigenvalue in the form $\Lambda_{1,2} = -h \pm j\omega_D$.

If the steps followed for the solution assumption $\varphi = e^{\Lambda t}$ are performed for the solution assumption $\varphi = e^{j\Lambda t}$, Holzer's Table 2 will be obtained ultimately. It will be called the *complex* (damped) Holzer table; its parameter is the eigenvalue in the form $\Lambda_{1,2} = jh \pm \omega_D$.† In the case of the undamped system, $s_i = u_i = 0$ in the tables and $h = 0$ in the eigenvalue. If in this case the eigenvalues $\Lambda = \pm j\omega_v$ and $\Lambda = \pm\omega_v$ are applied to Tables 1 and 2, respectively, the headings of both of these tables will be reduced to the heading of the *ordinary* Holzer table for undamped systems.

Table 1 Simple Holzer table for the calculation of damped linear systems; parameters: a) for free vibrations, $\Lambda = -h \pm j\omega_D$, and b) for forced vibrations, $(\Lambda) = j\Omega$

J_i	$-J_i\Lambda^2 - s_i\Lambda$	Φ_i	$(-J_i\Lambda^2 - s_i\Lambda)\Phi_i$	$\Sigma(-J_i\Lambda^2 - s_i\Lambda)\Phi_i$	$k_i + u_i\Lambda$	$\frac{\Sigma(-J_i\Lambda^2 - s_i\Lambda)\Phi_i}{(k_i + u_i\Lambda)}$
1	2	3	4 = 2 x 3	5	6	7 = 5/6

Thus, the statement about the eigenvalue as the parameter of the Holzer table in the case of free vibrations has been proved. It holds *generally*, i.e., for both damped and undamped systems (in the case of free vibrations). However, with the undamped systems both the eigenvalue and the tables themselves degenerate ($s_i = u_i = 0$ so that $h = 0$ and instead of ω_D there is ω_v). Consequently, in this special case of the general case the frequency may be (only practically!) considered as the parameter of the table. Theoretically, the parameter of the Holzer table in the case of free vibrations is only the eigenvalue irrespective of whether the system is damped or undamped.

Numerical Example

It is required that the eigenvalues of the first oscillatory mode ${}_1\Lambda_{1,2}$ be calculated for the torsional system of the following characteristics: $J_1 = 1$, $J_2 = 2$, $J_3 = 3$ (lb-in.-sec²), $s_1 = 0.12$, $s_2 = 0.20$, $s_3 = 0.32$ (lb-in.-sec.), $k_1 = \frac{1}{3}$, $k_2 = \frac{1}{2}$ (lb-in./rad), $u_1 = 0$, $u_2 = 0.04$ (lb-in.-sec).

Table 2 Complex Holzer table for the calculation of damped linear systems; parameters: a) for free vibrations, $\Lambda = jh \pm \omega_D$, and b) for forced vibrations, $(\Lambda) = \Omega$

J_i	$J_i\Lambda^2 - j\,s_i\Lambda$	Φ_i	$(J_i\Lambda^2 - j\,s_i\Lambda)\Phi_i$	$\Sigma(J_i\Lambda^2 - j\,s_i\Lambda)\Phi_i$	$k_i + ju_i\Lambda$	$\frac{\Sigma(J_i\Lambda^2 - j\,s_i\Lambda)\Phi_i}{(k_i + j\,u_i\Lambda)}$
1	2	3	4 = 2 x 3	5	6	7 = 5/6

The eigenvalues of the system can be evaluated by the trial-and-error procedure by means of either Table 1 or Table 2 (assumption of h and ω_D, i.e., of the eigenvalue Λ in correct form for the given table, calculation of the table, and drawing of the remainder-torque curve until the remainder torque becomes zero).

The resulting simple Holzer table for the eigenvalue of the first oscillatory mode ${}_1\Lambda_{1,2} = -0.0626773 \pm j\ 0.4960997$ is shown here as Table 3. (Hence, the damped natural frequency of the first mode is ${}_1\omega_D = 0.4960997$ sec^{-1}; the undamped natural frequency of the first mode, calculated separately, is ${}_1\omega_v = 0.5$ sec^{-1}.) It should be noticed that Table 3 for eigenvalue is "closed" (remainder torque is zero) as it has to be.

Table 3 Simple Holzer table calculated for ${}_1\Lambda_{1,2} = -0.0626773 \pm j\ 0.4960997$ (see numerical example)

J_i	$-J_i\Lambda^2 - s_i\Lambda$	Φ_i	$(-J_i\Lambda^2 - s_i\Lambda)\Phi_i$	$\Sigma(-J_i\Lambda^2 - s_i\Lambda)\Phi_i$	$k_i + u_i\Lambda$	$\frac{\Sigma(-J_i\Lambda^2 - s_i\Lambda)\Phi_i}{(k_i + u_i\Lambda)}$
1	0.2497078 + j 0.0026564	1.0	0.2497078 + j 0.0026564	0.2497078 + j 0.0026564	1/3	0.7491233 + j 0.0079693
2	0.4969084 + j 0.0251568	0.2508767 − j0.0079693	0.1248632 + j 0.0023513	0.3745710 + j 0.0050077	0.4974929 + j 0.0198440	0.7521221 − j 0.0199348
3	0.7466162 + j 0.0278133	−0.5012454 + j 0.0119655	−0.3745710 − j 0.0050077	0.0000000 + j 0.0000000		

In the case of forced vibrations of damped systems, the headings of both simple and complex Holzer tables remain as they are. In principle, the corresponding forms of the parameters hold in this case as well, except for $h = 0$. Thus, the simple Holzer table has to be calculated with the forcing frequency in the form $j\Omega$ and the complex one with that frequency in the form Ω, where Ω = forcing frequency.‡

The use of the damped-system Holzer tables for the calculation of both eigenvalues and eigenvectors of damped systems has been discussed fully in Refs. 1 and 2.

References

[1] Djodjo, B. A., "On vibrations of torsional systems with both external and internal torques acting on the system," Ph.D. Thesis, Dept. Mech. Eng., Univ. Belgrade (September 22, 1961).

[2] Djodjo, B. A., "On damped vibrations of discrete linear systems," to be published.

† Practically, the numerical values in both simple and complex Holzer tables are the same and complex because the parameters are complex in both cases.

‡ Hence, the incomplete view (that of the frequency as parameter of the Holzer table) is admissible in the case of forced vibrations calculated by the complex Holzer table. In all other cases (forced vibrations by the simple Holzer table, free vibrations by both tables), it induces the adoption of wrong procedures. On the other hand, only the eigenvalues of the simple Holzer table conform with the usual assumption that the "imaginary part of the eigenvalue is the natural frequency."

Systematic Matrix Calculation of Similarity Numbers

Victor J. Skoglund*
University of New Mexico, Albuquerque, N. Mex.

Because the dimensions of any macroscopic variable u_j are of the form

$$\{u_j\} = D_1^{a_{1j}}\, D_2^{a_{2j}}\, D_3^{a_{3j}}\, D_4^{a_{4j}} \tag{1}$$

where D_1 is mass, D_2 length, D_3 time, D_4 temperature, and a_{ij} a positive or negative integer or fraction, the dimensions of a set of variables $u_1, \ldots u_n$ may be represented by a dimensional matrix $\|a_{ij}\|$, where i indicates the dimension or row number and j indicates the variable or column number. By definition, a similarity number is a nondimensional product of variables of the form

$$N = u_1^{e_1}\, u_2^{e_2} \ldots u_n^{e_n} \tag{2}$$

Received December 17, 1962.

* Professor, Mechanical Engineering Department. Member AIAA.

To satisfy the nondimensional requirement of N, e_j must satisfy a set of linear, homogeneous equations

$$\sum_{j=1}^{n} a_{ij}\, e_j = 0$$

In solving these equations for e_j in a systematic manner, it is convenient to transform the dimensional matrix to a unitized form by the following operations: 1) interchanging rows, 2) multiplying a row by a nonzero constant, 3) adding one row to another, and 4) if necessary, interchanging columns.

As a result, the number of rows in the unitized form may be less than in the dimensional matrix. A unitized matrix with four rows is of the form

$$\|b_{ij}\| = \left\|\begin{matrix} 1 & 0 & 0 & 0 & \cdots & b_{1n} \\ 0 & 1 & 0 & 0 & \cdots & b_{2n} \\ 0 & 0 & 1 & 0 & \cdots & b_{3n} \\ 0 & 0 & 0 & 1 & \cdots & b_{4n} \end{matrix}\right\| \tag{3}$$

Since the foregoing row and column operations also may be applied to the equations of exponents without affecting their results, the unitized matrix may be used to form a new set of equations:

$$\sum_{j=1}^{n} b_{ij}\, e_j = 0$$

By matrix theory, the number of independent equations of such a set equals the number of nonzero rows in the unitized matrix. That number also equals the rank of the matrix. Therefore, with r nonzero rows, $e_1, \ldots e_r$ may be considered dependent, and $e_{r+1}, \ldots e_n$ may be considered independent. The exponents e_{jk} of a single similarity number N_k may be obtained by letting one of the independent exponents $e_{r+k} = 1$ and letting the other independent exponents equal zero. The result is

$$N_k = u_1^{e_{1k}}\, u_2^{e_{2k}} \ldots u_r^{e_{rk}}\, u_{r+k} \tag{4}$$

The possibilities for k are $k = 1, \ldots (n - r)$. Therefore, a principle of dimensional analyses is that the number of similarity numbers in a set is $s = n - r$, where n and r are the number of columns and rows in the unitized dimensional matrix.

Influence of Constant Disturbing Torques on the Motion of Gravity-Gradient Stabilized Satellites

T. B. Garber*
Rand Corporation, Santa Monica, Calif.

THE conditions under which a body subject to gravitational-gradient torques will perform stable oscillations about an equilibrium point are well known.[1, 2] However, in most practical cases, torques other than those due to the gravity gradient will act on the satellite.[3] The purpose of this note is to demonstrate the effect of a constant disturbing torque upon the transient response of a gravity-gradient stabilized body.

Received by IAS November 30, 1962. Any views expressed in this paper are those of the author. They should not be interpreted as reflecting the views of the Rand Corporation or the official opinion or policy of any of its governmental or private research sponsors. Papers are reproduced by the Rand Corporation as a courtesy to members of its staff.

* Physical Scientist, Electronics Department. Member AIAA.

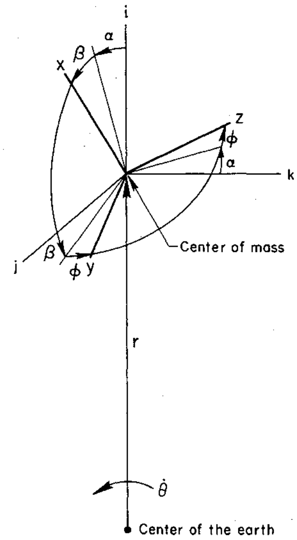

Fig. 1 Orientation of body axes with respect to orbital local horizontal coordinates

It is assumed that the satellite is on a circular orbit. Thus the oblateness and other asymmetries of the earth are neglected. The only torques that act on the body are those due to the gravity gradient and the constant disturbance.

Euler's rotational equations of motion are

$$\dot{\omega}_x - R_x\omega_y\omega_z = (M_x/I_x)_G + (M_x/I_x)_D \tag{1a}$$

$$\dot{\omega}_y - R_y\omega_z\omega_x = (M_y/I_y)_G + (M_y/I_y)_D \tag{1b}$$

$$\dot{\omega}_z - R_z\omega_x\omega_y = (M_z/I_z)_G + (M_z/I_z)_D \tag{1c}$$

where

$$R_x = (I_y - I_z)/I_x$$

$$R_y = (I_z - I_x)/I_y$$

$$R_z = (I_x - I_y)/I_z$$

The subscript G refers to the gradient torque, whereas D indicates a disturbing torque. From the form of Eqs. (1a–1c), it can be seen that the $x\ y\ z$ axes are central principal axes.

The orientation of the body with respect to the local horizontal coordinates is defined by the angles α, β, and φ (see Fig. 1). In terms of the orientation angles and their derivatives, the body angular rates are

$$\omega_x = \dot{\varphi} + (\dot{\alpha} + \dot{\theta}) \sin\beta \tag{2a}$$

$$\omega_y = \beta \sin\varphi + (\dot{\alpha} + \dot{\theta}) \cos\beta \cos\varphi \tag{2b}$$

$$\omega_z = \beta \cos\varphi - (\dot{\alpha} + \dot{\theta}) \cos\beta \sin\varphi \tag{2c}$$

where $\dot{\theta}$ is the angular rate of the local horizontal axes due to the orbital motion. Finally, the gravitational-gradient torques are[4]

$$(M_x/I_x)_G = -3\dot{\theta}^2 R_x(\sin\alpha \cos\varphi + \cos\alpha \sin\beta \sin\varphi) \times (\sin\alpha \sin\varphi - \cos\alpha \sin\beta \cos\varphi) \tag{3a}$$

$$(M_y/I_y)_G = -3\dot{\theta}^2 R_y \cos\alpha \cos\beta(\sin\alpha \cos\varphi + \cos\alpha \sin\beta \sin\varphi) \tag{3b}$$

$$(M_z/I_z)_G = -3\dot{\theta}^2 R_z \cos\alpha \cos\beta(\sin\alpha \sin\varphi - \cos\alpha \sin\beta \cos\varphi) \tag{3c}$$

Consider the case in which there is a steady state pitch angle. Such a condition might arise physically due to residual drag forces acting in conjunction with a center-of-mass, center-of-pressure separation. Thus a steady-state value of α develops until the gradient torque is equal to the disturbance in magnitude.